

Assignment 8.

This homework is due *Thursday*, October 29.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. QUICK REMINDER

Simple Approximation Lemma. Let $f : E \rightarrow \mathbb{R}$ be measurable. Assume f is bounded on E . Then for each $\varepsilon > 0$, there are simple functions $\varphi_\varepsilon, \psi_\varepsilon$ defined on E such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon, \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

Egoroff's Theorem. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f . Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \rightarrow f \text{ uniformly on } F, \text{ and } m(E \setminus F) < \varepsilon.$$

Lusin's Theorem. Let $f : E \rightarrow \mathbb{R}$ be measurable. Then for each $\varepsilon > 0$, there is a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ and a closed set F contained in E such that

$$f = g \text{ on } F, \text{ and } m(E \setminus F) < \varepsilon.$$

Lebesgue integral of bounded function. For a bounded function f on a set E of finite measure, Lebesgue integral $\int_E f$ is defined as the common value of $\sup\{\int_E \varphi \mid \varphi \leq f, \varphi \text{ simple}\}$ and $\inf\{\int_E \psi \mid \psi \geq f, \psi \text{ simple}\}$, if the latter two are equal.

2. EXERCISES

- (1) (3.2.21-) For a sequence $\{f_n\}$ of measurable functions with common domain E , show that $\inf\{f_n\}$ and $\sup\{f_n\}$ are measurable. (*Hint:* Express sup and inf as limits of appropriate sequences; or directly use the definition of a measurable function.)
- (2) In class Lusin's theorem was proved for the case $m(E) < \infty$. Finish the proof in the case $m(E) = \infty$. (You can take the case $m(E) < \infty$, and anything in its proof, for granted.)
- (3) (3.3.26) Let $E = [0, 1]$. Lusin's theorem provides that a measurable $f : E \rightarrow \mathbb{R}$ equals to some continuous g at points of a set $F \subset E$. Must f itself be continuous at any point (as a function on E)?
- (4) (3.3.27) Show that the conclusion of Egoroff's theorem is false if we drop the assumption that the domain E has finite measure. (*Hint:* Take a function f that is zero on \mathbb{R}_- and nonzero on \mathbb{R}_+ , consider $\{f(x - n)\}$.)

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- (5) (3.3.31) Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued f pointwise on E . Show that $E = E_0 \cup \bigcup_{k=1}^{\infty} E_k$, where all sets E_0, E_1, \dots are measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 0$, and $m(E_0) = 0$.
- (6) (4.1.6+) Prove that a continuous function on a closed interval $[a, b]$ ($a, b \in \mathbb{R}$) is Riemann integrable. (*Hint*: Use uniform continuity to put an upper bound on the difference between upper and lower Darboux sums.)
- (7) (4.2.9) Let E have measure zero. Show that if f is a bounded function on E , then f is measurable and $\int_E f = 0$.
- (8) Let f be a simple function on a set E of finite measure. Then two definitions of Lebesgue integral apply to f : the Lebesgue integral of a simple function, and the Lebesgue integral of a bounded function. Show that these definitions agree on f , i.e. they both give the same value of $\int_E f$.